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A theorem of instability by linear approximation for a one-dimensional non-linearly elastic body[☆]

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Abstract

On the basis of and in a development of the ideas and results of A.A. Movchan (Sr.), that extend to continuous bodies the definitions and main fundamental theorems of Lyapunov on stability and instability, a criterion for instability of the equilibrium position of a one-dimensional non-linearly elastic body subject to potential external forces is established. For the specified simplest type of continuous elastic system (which possesses, however, a number of fundamental properties of continuous elastic systems including unboundedness of the operator of linear approximation and discreteness of its spectrum) a theorem of instability by linear approximation is stated and proved. The method of proof is a version of Persidskii's sector method.

Since the publication (over a hundred years ago) of famous Lyapunov's famous book "The General Problem of the Stability of Motion", research in the field relating to systems with a finite number of degrees of freedom, directly or indirectly employ (and also develop) his results, ideas and methods, which have proved to be extremely fruitful.

As for research on the stability of continuous systems (systems with an infinite number of degrees of freedom), in most cases (including previous papers by the author, too) they ignore both Lyapunov's theory of stability and its existing (but insufficiently well-known) generalizations to systems of this type.[1–9] As a rule the research mentioned is based upon static or dynamic "criteria", well-known in this branch of mechanics, whose adequacy to the rigorous theory and in general to the notion of stability formed in the mechanics of systems with a finite number of degrees of freedom is far from being indisputable, which gives reasons to doubt its results and conclusions. The alternative to all of this is perhaps to consider the problems of the stability and instability of continuous systems on the basis of the corresponding rigorous theory which is what will be done in this paper; specifically, a generalization of Lyapunov's theory worked out by A.A. Movchan (Sr.)[3–5] is used.

The simplest and clearest (from the standpoint of a mathematical description) type of continuous systems is that of elastic systems. Conservative elastic systems (including continuous ones) are ideally suited for using Lyapunov's direct method in the case of stability: the obvious candidate for the role of Lyapunov's function (functional) is the total energy of the system, which is preserved by virtue of the equations of motion. In just this way Movchan gave an exact meaning to and proved (within the framework of his generalization of Lyapunov's theory) Lagrange's theorem for continuous elastic bodies: the sufficient condition for stability is that the total potential energy should be positive definite with respect to some specially selected norm.[5]

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As for instability of the same conservative systems, this question is much more complicated, and there are no results that provide and substantiate some effective criteria for the instability of continuous bodies. Note that the proofs of theorems on criteria for instability are much more complicated than the proofs of theorems on criteria for stability even in the case of finite-dimensional (in particular, conservative) systems. The only simple and natural case a very special case when the total potential energy (measured from the equilibrium state) is negative definite. Moreover Chetayev proved (by means of his own well-known theorem) the instability for negative values of the potential energy in the quite specific (one can say, an odd) case, when the potential energy is a homogeneous function of a certain degree. The point is that for linear systems (for which the potential energy is quadratic) this result is not of great interest, whereas non-linear systems of this type are quite unusual or even unreal (the presence of a sign-variable quadratic form that represents the potential energy of a linearized system, and higher order terms is typical). Movchan[5] presents a version of Chetayev's proof (with minimal changes) in the case of "appropriate" elastic continuous bodies, but due to some features of the non-linear theory of elasticity these simply do not exist.

In the general case, when the potential energy of a finite-dimensional system is not negative definite but can take negative values in the vicinity of the equilibrium position, the conservatism is by no means involved in the standard proof of instability, but rests on Lyapunov's general theorem of instability by linear approximation. However, Lyapunov's proof and its modifications are based on the finite dimensionality of a space. As for the two versions of a theorem of instability by linear approximation presented in Ref. 10, although they relate to equations in infinite-dimensional spaces, according to the method of proof they are based on the assumption of the boundedness of the linear operator, which makes it impossible to apply them directly to the problem of the instability of continuous elastic systems, where the operator of the linearized problem is always unbounded (this is clearly pointed out in the preface to the book mentioned).

Thus, for continuous systems, including conservative ones, there are obviously no proven results of the type of Lyapunov's theorem of instability by linear approximation. This is why the present paper is timely: we formulate and prove a theorem of instability in a linear approximation at least for the simplest non-linear continuous conservative system, specifically for a one-dimensional non-linearly elastic body with fixed ends. The method of proof employed is quite different from that of the classical Lyapunov's proof and from that used in the proofs of corresponding theorems in Ref. 10; it is a version of Persidskii's sector method.[11–13] The main condition for the applicability of the proposed version of the sector method to prove the theorem stated, is the discreteness of spectrum of the linear operator, which also occurs in the general case for continuous elastic systems.

1. The exact and linearized equations of motion for a one-dimensional elastic body

1.1. The main assumptions regarding the coefficients

1.1.1. The energy of the body

We will consider a one-dimensional elastic body occupying the section $[0, l]$ in the reference configuration; values of variable $x \in [0, l]$ correspond to points of the body. Displacement of the body will be called the scalar-valued function $u(x, t)$, where t is the time, which is assumed to be twice continuously differentiable with respect to both arguments. The ends of the one-dimensional body are assumed to be fixed:

$$u(0, t) = u(l, t) = 0. \tag{1.1}$$

Denoting the derivative with respect to t for constant x (i.e. the material time derivative) by a dot over a symbol, and that with respect to x for constant t by a subscript x, we take for the body the following equation of motion (the momentum equation):

$$\mu(x)\ddot{u}(x, t) = \left(p(x)u_x(x, t) + \frac{1}{2}\frac{\partial\psi}{\partial u_x}(x, u_x(x, t))\right)_x + q(x)u(x, t) + \frac{1}{2}\frac{\partial\varphi}{\partial u}(x, u(x, t)). \tag{1.2}$$

By supposition, the functions $\mu(x)$ and $q(x)$ are continuous and $p(x)$ is differentiable in the interval $[0, l]$ and all the three functions are positive. The function $\psi(x, u_x)$ posseses, by assumption, the following properties: $\psi(x, 0) = 0$, $\psi(x, u_x)$ is twice differentiable with respect to both arguments, and the following inequality holds

$$\left|\frac{\partial\psi}{\partial u_x}(x, u_x)\right| \le C_1|u_x|^{1+\sigma_1}, \tag{1.3}$$

where $C_1 > 0$, $\sigma_1 > 0$ are constants. From inequality (1.3) it follows that

$$\frac{\partial\psi}{\partial u_x}(x,0) = 0.$$

Additionally from inequality (1.3) we can readily derive the inequality

$$|\psi(x,u_x)| \le \frac{C_1}{2+\sigma_1}|u_x|^{2+\sigma_1}. \tag{1.4}$$

The supposed properties of the function $\varphi(x, u)$ are similar: $\varphi(x, 0) = 0$, it is differentiable with respect to the second argument, and the following inequality holds

$$\left|\frac{\partial\varphi}{\partial u}(x,u)\right| \le C_2|u|^{1+\sigma_2}; \quad C_2 > 0; \quad \sigma_2 > 0 \tag{1.5}$$

from whence it follows that

$$\frac{\partial\varphi}{\partial u}(x,0) = 0.$$

Moreover we have the inequality

$$|\varphi(x,u)| \le \frac{C_2}{2+\sigma_2}|u|^{2+\sigma_2}. \tag{1.6}$$

If we set equal to zero in Eq. (1.2) terms containing the functions ψ and φ, then we obtain the linearized equation

$$\mu(x)\ddot{u}(x,t) = (p(x)u_x(x,t))_x + q(x)u(x,t) \tag{1.7}$$

on the right hand side of which there is an operator opposite in sign to the Sturm–Liouville operator.

Note that the mechanical system generating the momentum Eq. (1.2), has a two-fold interpretation: (1) a one-dimensional elastic body, and hence $u(x, t)$ is a one-dimensional (longitudinal) displacement; (2) a string, in this case $u(x, t)$ is the transverse displacement. In both cases $\mu(x)$ is the density per unit length, the term depending on u_x is the elastic stress, while that depending on u is the distributed external potential force (in the case in hand, due to the positiveness of $q(x)$, the one repelling from the equilibrium position and causing instability).

Denoting integration over the interval $[0, l]$ by angle brackets, and also denoting the velocity field $\dot{u}(x, t)$ by $v(x, t)$, we can write the total energy of the body $e\{u, v\}$, which is obviously preserved during its motion:

$$e\{u,v\} = \frac{1}{2}\langle\mu v^2\rangle + \frac{1}{2}\langle p u_x^2 + \psi\rangle - \frac{1}{2}\langle q u^2 + \varphi\rangle; \quad \dot{e}\{u,v\} = 0. \tag{1.8}$$

We will consider the fields of displacements, velocities and other continuous functions in the interval $[0, l]$ as elements of a Hilbert space with scalar product

$$u\tilde{u} \equiv \langle\mu u\cdot\tilde{u}\rangle. \tag{1.9}$$

Introducing the linear operator A (we will call it the generalized Sturm–Liouville operator) and the non-linear operator f by the equalities

$$Au \equiv -\frac{1}{\mu}(pu_x)_x - \frac{1}{\mu}qu, \quad f\{u\} \equiv \frac{1}{2\mu}\left(\frac{\partial\psi}{\partial u_x}\right)_x + \frac{1}{2\mu}\frac{\partial\varphi}{\partial u} \tag{1.10}$$

we will write the equation of motion as a system of two equations

$$\dot{u} = v, \quad \dot{v} = -Au + f\{u\}. \tag{1.11}$$

It is obvious that the pair of zero fields $u(x, t) = 0$, $v(x, t) = 0$ is the zero solution (in other words, the equilibrium position) of the system of Eq. (1.11). It is the instability of this equilibrium position that is under consideration in this paper.

Note that the operator A is symmetric with respect to the scalar product on the functions that are twice differentiable and vanish at the ends of interval:

$$\tilde{u} \cdot Au = u \cdot A\tilde{u}. \tag{1.12}$$

The linearized system of equations of motion takes the form

$$\dot{u} = \upsilon, \quad \dot{\upsilon} = -Au, \tag{1.13}$$

and the energy of the body can be represented as follows:

$$e\{u, \upsilon\} = \frac{1}{2}\upsilon \cdot \upsilon + \frac{1}{2}u \cdot Au + \frac{1}{2}\Psi\{u\}, \quad \Psi\{u\} \equiv \langle\psi\rangle - \langle\varphi\rangle. \tag{1.14}$$

2. Well-known spectral properties of the operator A

2.1. Definition of the operator B and upper and lower estimates for its quadratic form

The spectral and related properties of the generalized Sturm–Liouville operator are well-known. We will list the ones that are used below. The operator A is semibounded below, it has a real discrete spectrum $\{a_n\}$ ($n = 1, 2, \ldots, \infty$), where $\lim a_n = +\infty$ as $n \to \infty$. We will assume that the eigenvalues are numbered in non-decreasing order: $a_1 \le a_2 \le \ldots$. There is an orthonormal system of eigenfunctions $\{g_n\}$:

$$Ag_n = a_n g_n, \quad g_i \cdot g_j = \delta_{ij}. \tag{2.1}$$

Note that the span of any finite set of eigenfunctions say, span$(g_1, \ldots, g_n)$, is an invariant subspace of the operator, the same being true for its orthogonal (in the sense of the introduced scalar product) supplement span$(g_1, \ldots, g_n)^{\perp}$.

By the well-known variational description of the eigenvalues and eigenfunctions the following inequalities hold:

$$u \cdot Au \ge a_{n+1} u \cdot u, \quad \forall u \in \operatorname{span}(g_1, \ldots g_n)^{\perp}. \tag{2.2}$$

Bearing in mind the feasibility in principle of some further generalizations, we do not exclude the possible existence of multiple eigenvalues (of finite multiplicity) of the operator A.

We will assume that the first m eigenvalues of the operator A are negative, the next m' eigenvalues are equal to zero, and all the remaining eigenvalues are positive:

$$\begin{aligned} &a_i < 0, \quad i = 1, \ldots, m \\ &a_j = 0, \quad j = m+1, \ldots, m+m' \\ &a_k > 0, \quad k = m+m'+1, \ldots, \infty. \end{aligned} \tag{2.3}$$

For convenience we will introduce the dyad operators, in conventional way corresponding to the scalar product (1.9):

$$(g \otimes h)u \equiv g(h \cdot u). \tag{2.4}$$

From the specified operator A, we construct a positive definite operator B having the same eigenfunctions and a strictly positive spectrum. Suppose $b_0 > 0$ is, for now, an arbitrary positive number, for which certain restrictions are to be taken later. We specify the operator B by the equality:

$$B \equiv \sum_{i=1}^{m} 2|a_i| g_i \otimes g_i + \sum_{j=m+1}^{m+m'} b_0 g_j \otimes g_j + A. \tag{2.5}$$

It is obvious that

$$\begin{aligned} &Bg_i = b_i g_i, \quad b_i = |a_i| > 0, \quad i = 1, \ldots, m \\ &Bg_j = b_j g_j, \quad b_j = b_0 > 0, \quad j = m+1, \ldots, m+m' \\ &Bg_k = b_k g_k, \quad b_k = a_k > 0, \quad k = m+m'+1, \ldots, \infty. \end{aligned} \tag{2.6}$$

Thus, the operator B differs from the operator A by a finite-dimensional operator that is positive definite in its own subspace span$(g_1, \ldots, g_{m+m'})$; in the infinite-dimensional subspace span$(g_1, \ldots, g_{m+m'})^\perp$ the operators B and A act identically.

Using the fact that for each operator both the finite-dimensional subspace span$(g_1, \ldots, g_{m+m'})$ and the infinite-dimensional subspace span$(g_1, \ldots, g_{m+m'})^\perp$ are invariant, and taking into account inequality (2.2) it is easy to prove the lower estimate for quadratic form of the operator B, refining the character of its positive definitness:

$$u \cdot Bu \geq b_{\min} u \cdot u; \quad b_{\min} \equiv \min(|a_m|, b_0, a_{m+m'+1}) > 0. \tag{2.7}$$

However, for our further analysis we need two more estimates (upper and lower ones) for the same quadratic form in terms of the quantity $\langle u_x^2 \rangle$. We obtain these estimates, having first introduced the following notation

$$\underline{\chi} \equiv \min_x \chi(x), \quad \bar{\chi} \equiv \max_x \chi(x); \quad \chi = \mu, p, q; \quad \Theta \equiv \bar{q}/\underline{\mu}.$$

We will begin with a simpler estimate, namely the upper one:

$$u \cdot Bu \leq \max(2|a_1|, b_0) u \cdot u + u \cdot Au \leq \bar{\mu} \max(2|a_1|, b_0) \langle u^2 \rangle + \langle p u_x^2 \rangle \leq$$
$$\leq (\bar{\mu} l^2 \max(2|a_1|, b_0) + \bar{p}) \langle u_x^2 \rangle \equiv \bar{C} \langle u_x^2 \rangle.$$

To obtain the lower estimate, we first consider the operators

$$\tilde{A} = A + \Theta I, \quad \tilde{B} = B + \Theta I; \quad Iu = u, \quad \tilde{B} = \tilde{A} + (B - A).$$

The operator $\tilde{B}$ has the same eigenfunctions as B, and for its quadratic form we have the lower estimate

$$u \cdot \tilde{B} u \geq u \cdot \tilde{A} u = \langle p u_x^2 \rangle + \langle (\Theta\mu - q) u^2 \rangle \geq \underline{p} \langle u_x^2 \rangle. \tag{2.8}$$

On the other hand we have

$$u \cdot \tilde{B} u = u \cdot Bu + \frac{\Theta}{b_{\min}} b_{\min} u \cdot u \leq \left(1 + \frac{\Theta}{b_{\min}}\right) u \cdot Bu \equiv N u \cdot Bu. \tag{2.9}$$

From the relations (2.8) and (2.9) we obtain

$$\underline{C} \langle u_x^2 \rangle \equiv \frac{\underline{p}}{N} \langle u_x^2 \rangle \leq \frac{1}{N} u \cdot \tilde{B} u \leq u \cdot Bu.$$

Hence, we have both upper and lower estimates

$$\underline{C} \langle u_x^2 \rangle \leq u \cdot Bu \leq \bar{C} \langle u_x^2 \rangle. \tag{2.10}$$

3. The space of "displacement – velocity" pairs (phase space)

3.1. The equivalent systems of equations (non-linear and linearized) in phase space; the linear operator of the linearized system and the special scalar product

3.1.1. The special norm

We will introduce the phase (real) linear space, i.e. the space of pairs

$$\mathbf{w}(x, t) = \left\| \begin{matrix} u(x, t) \\ v(x, t) \end{matrix} \right\|; \quad q_1 \mathbf{w}_1 + q_2 \mathbf{w}_2 \equiv \left\| \begin{matrix} q_1 u_1(x, t) + q_2 u_2(x, t) \\ q_1 v_1(x, t) + q_2 v_2(x, t) \end{matrix} \right\|.$$

Then the non-linear system of Eq. (1.11) will take the form

$$\dot{\mathbf{w}} = \mathcal{A}\mathbf{w} + \mathbf{f}\{\mathbf{w}\}; \quad \mathcal{A}\mathbf{w} \equiv \mathcal{A} \left\| \begin{matrix} u \\ v \end{matrix} \right\| = \left\| \begin{matrix} v \\ -Au \end{matrix} \right\|, \quad \mathbf{f}\{\mathbf{w}\} = \left\| \begin{matrix} 0 \\ f\{u\} \end{matrix} \right\|. \tag{3.1}$$

The operator $\mathscr{A}$ acting on the space of pairs, can be represented in the form of a pseudo-matrix

$$\mathscr{A} = \begin{Vmatrix} 0 & I \\ -A & 0 \end{Vmatrix},$$

where I is the identity operator in the original functional space.

The linearized system of Eq. (1.13) is a result of dropping the term $\mathbf{f}\{\mathbf{w}\}$

$$\dot{\mathbf{w}} = \mathscr{A}\mathbf{w}. \tag{3.2}$$

Obviously the properties of the operator $\mathscr{A}$ are generated by the properties of the operator A. We will consider some of them. First of all we note that every proper one-dimensional subspace of the operator A spanned on the eigenfunction g_i, generates a two-dimensional invariant subspace of the operator $\mathscr{A}$ spanned on the pair of vectors $\mathbf{g}_i^u \equiv \begin{Vmatrix} g_i \\ 0 \end{Vmatrix}$ and $\mathbf{g}_i^v \equiv \begin{Vmatrix} 0 \\ g_i \end{Vmatrix}$. Indeed, let

$$\mathbf{w} = u_i\mathbf{g}_i^u + v_i\mathbf{g}_i^v = \begin{Vmatrix} u_i g_i \\ v_i g_i \end{Vmatrix}. \tag{3.3}$$

Then

$$\mathscr{A}\mathbf{w} = \begin{Vmatrix} v_i g_i \\ -u_i A g_i \end{Vmatrix} = \begin{Vmatrix} v_i g_i \\ -a_i u_i g_i \end{Vmatrix} = v_i\mathbf{g}_i^u - a_i u_i \mathbf{g}_i^v. \tag{3.4}$$

However, some features of the action of the operator $\mathscr{A}$ on these two-dimensional subspaces are different depending on the sign of the eigenvalue a_i.

If $a_i < 0$, we have

$$-a_i = |a_i| \equiv \alpha_i^2, \quad \alpha_i > 0 \tag{3.5}$$

$$\boldsymbol{\gamma}_i^{\pm} \equiv \mathbf{g}_i^u \pm \alpha_i \mathbf{g}_i^v, \quad \mathscr{A}\boldsymbol{\gamma}_i^{\pm} = \pm\alpha_i\boldsymbol{\gamma}_i^{\pm}. \tag{3.6}$$

Hence, in that case the two-dimensional invariant subspace splits into the direct sum of two one-dimensional subspaces spanned on the eigenvectors $\boldsymbol{\gamma}_i^+$ (with the eigenvalues α_i) and $\boldsymbol{\gamma}_i^-$ (with the eigenvalues $-\alpha_i$). Note that the exponential in time solutions of the linearized system

$$\mathbf{w}_i^{\pm}(x, t) = u_{i0}^{\pm} e^{\pm\alpha_i t} \boldsymbol{\gamma}_i^{\pm}, \tag{3.7}$$

where u_{i0}^+, u_{i0}^- are arbitrary numbers, correspond to these eigenspaces. The solutions with increasing exponent indicate the instability of the linearized system (no matter how that instability is defined), but by themselves signify nothing regarding the instability of the non-linear system.

If $a_i = 0$, then for $\mathbf{w}$ (3.3) we have

$$\mathscr{A}\mathbf{w} = v_i\mathbf{g}_i^u; \quad \mathscr{A}\mathbf{g}_i^u = 0, \quad \mathscr{A}\mathbf{g}_i^v = \mathbf{g}_i^u. \tag{3.8}$$

The solutions of the linearized system here take the form

$$\mathbf{w}_i(x, t) = (u_{i0} + v_{i0}t)\mathbf{g}_i^u + v_{i0}\mathbf{g}_i^v$$

and for $v_{i0} \neq 0$ also move away from the equilibrium state.

If $a_i > 0$, then there is a couple of pure imaginary roots of the characteristic equation, there being no one-dimensional invariant subspaces (we recall that all the spaces introduced are real); the corresponding solutions are oscillatory (linear combinations of sines and cosines):

$$\begin{aligned}&\omega_i \equiv \sqrt{a_i}, \quad \mathcal{A}\mathbf{g}_i^u = -\omega_i^2\mathbf{g}_i^v; \quad \mathcal{A}\mathbf{g}_i^v = \mathbf{g}_i^u\\ &\mathbf{w}_i(x,t) = \left(u_{i0}\cos\omega_i t + \frac{v_{i0}}{\omega_i}\sin\omega_i t\right)\mathbf{g}_i^u + (v_{i0}\cos\omega_i t - \omega_i u_{i0}\sin\omega_i t)\mathbf{g}_i^v.\end{aligned} \tag{3.9}$$

These solutions remain in the vicinity of the equilibrium position.

We will introduce in the phase space a special scalar product related to the operator B, and related via it to the operator A and its eigenfunctions:

$$\mathbf{w} \bullet \mathbf{w}' \equiv \left\| \begin{matrix} u \\ v \end{matrix} \right\| \bullet \left\| \begin{matrix} u' \\ v' \end{matrix} \right\| \equiv u \cdot Bu' + v \cdot v'. \tag{3.10}$$

All the properties of the scalar product obviously follow from the previously proved properties of the operator B and its quadratic form. Note that the scalar square of $\mathbf{w}$ contains as one of the terms the doubled total energy of the linearized system:

$$\begin{aligned}&B = -2A^- + b_0 I^0 + A = -A^- + b_0 I^0 + A^+; \quad A^- \equiv \sum_{i=1}^{m} a_i g_i \otimes g_i\\ &I^0 = \sum_{i=m+1}^{m+m'} g_i \otimes g_i; \quad A = A^- + A^+.\end{aligned}$$

We have

$$\mathbf{w} \bullet \mathbf{w} = u \cdot Au + v \cdot v - 2u \cdot A^- u + b_0 u \cdot I^0 u, \tag{3.11}$$

where the additional terms are related only to the subspace span$(\mathcal{W}_1, \ldots, \mathcal{W}_{m+m'})$ corresponding to the non-positive eigenvalues; here

$$\mathcal{W}_i \equiv \mathrm{span}(\mathbf{g}_i^u, \mathbf{g}_i^v). \tag{3.12}$$

For what follows we need to decompose the operator $\mathcal{A}$ into symmetric and skew-symmetric parts with respect to the introduced scalar product in phase space. For this purpose we find $\mathcal{A}^T\mathbf{w}$:

$$\mathbf{w}'' \equiv \mathcal{A}^T\mathbf{w} \Rightarrow \mathbf{w}' \bullet \mathcal{A}^T\mathbf{w} \equiv \mathbf{w}' \bullet \mathbf{w}'' = \mathbf{w} \bullet \mathcal{A}\mathbf{w}', \quad \forall \mathbf{w}, \mathbf{w}'$$

$$\mathcal{A}\mathbf{w}' = \left\| \begin{matrix} v' \\ -Au' \end{matrix} \right\|, \quad \mathbf{w} \bullet \mathcal{A}\mathbf{w}' = u \cdot Bv' - v \cdot Au' = u' \cdot (-Av) + v' \cdot Bu = u' \cdot Bu'' + v' \cdot v''.$$

Then

$$v'' = Bu, \quad Bu'' = -Av. \tag{3.13}$$

On each of the three subspaces

$$\mathrm{span}(g_1, \ldots g_m) \equiv U^-, \quad \mathrm{span}(g_{m+1}, \ldots g_{m+m'}) \equiv U^0, \quad \mathrm{span}(g_1, \ldots g_{m+m'})^{\perp} \equiv U^+$$

the second Equation of (3.13) takes the form

$$-A(u'')^- = -Av^- \Rightarrow (u'')^- = v^-, \quad b_0(u'')^0 = 0 \Rightarrow (u'')^0 = 0, \quad A(u'')^+ = -Av^+ \Rightarrow (u'')^+ = -v^+.$$

Hence it follows, that

$$\mathcal{A}^T\mathbf{w} \equiv \mathbf{w}'' = \left\| \begin{matrix} u'' \\ v'' \end{matrix} \right\| = \left\| \begin{matrix} v^- - v^+ \\ Bu \end{matrix} \right\| = \left\| \begin{matrix} v^- \\ -Au^- \end{matrix} \right\| + \left\| \begin{matrix} 0 \\ b_0 u^0 \end{matrix} \right\| - \left\| \begin{matrix} v^+ \\ -Au^+ \end{matrix} \right\| \tag{3.14}$$

$$\mathcal{A}^s = \frac{1}{2}(\mathcal{A} + \mathcal{A}^T), \quad \mathcal{A}^s\mathbf{w} \equiv \mathcal{A}^s \begin{Vmatrix} u \\ \upsilon \end{Vmatrix} = \begin{Vmatrix} \upsilon^- + \upsilon^0/2 \\ -Au^- + b_0u^0/2 \end{Vmatrix} \tag{3.15}$$

$$\mathcal{A}^a = \frac{1}{2}(\mathcal{A} - \mathcal{A}^T), \quad \mathcal{A}^a\mathbf{w} = \begin{Vmatrix} \upsilon^+ + \upsilon^0/2 \\ -Au^+ - b_0u^0/2 \end{Vmatrix}. \tag{3.16}$$

From relations (3.15) and (3.16) it is clear that the operators $\mathcal{A}^s$ and $\mathcal{A}^a$ are "geared" only in the subspace $\mathcal{W}^0$ induced by the zero subspace of the operator A, and hence they are "ungeared" when it is absent. In the finite-dimensional subspace $\mathcal{W}^-$ (correspondingly, in the infinite-dimensional subspace $\mathcal{W}^+$) induced by the eigenfunctions of the operator A with negative (correspondingly, positive) eigenvalues, the operator $\mathcal{A}$ is symmetric (correspondingly, skew-symmetric). In the finite-dimensional subspace $\mathcal{W}^0$ the operator $\mathcal{A}$ has both symmetric and skew-symmetric parts. For each of the operators $\mathcal{A}$, $\mathcal{A}^s$ and $\mathcal{A}^a$ the subspaces $\mathcal{W}^-$, $\mathcal{W}^0$ and $\mathcal{W}^+$ are invariant; for the restrictions of the operators mentioned to these subspaces we give their notation an additional superscript taking the values −, 0 or +, respectively. In this case

$$\mathcal{A}^- = \mathcal{A}^{s-}, \quad \mathcal{A}^{a-} = 0; \quad \mathcal{A}^+ = \mathcal{A}^{a+}, \quad \mathcal{A}^{s+} = 0; \quad \mathcal{A}^0 = \mathcal{A}^{s0} + \mathcal{A}^{a0}. \tag{3.17}$$

For our subsequent analysis we introduce in the phase space a special norm that majorizes the one generated by the scalar product:

$$|\mathbf{w}|^2 \equiv \bar{C}l\sup(u_x^2) + \upsilon \cdot \upsilon \tag{3.18}$$

$$\underline{C}\langle u_x^2\rangle + \upsilon \cdot \upsilon \le \mathbf{w} \bullet \mathbf{w} = u \cdot Bu + \upsilon \cdot \upsilon \le \bar{C}\langle u_x^2\rangle + \upsilon \cdot \upsilon \le |\mathbf{w}|^2, \tag{3.19}$$

where supremum is taken for $x \in [0, l]$. The instability established here is first of all just the one with respect to the norm (3.18), although in what follows we will derive from this one an instability of a somewhat different kind (more fitting the traditions and the spirit of the theory of elasticity and the mechanics of solids, in general).

4. The first definition of instability

4.1. The Lyapunov functional and the proof of instability

We will consider the instability of the equilibrium position (the zero solution) of system (1.11) with zero boundary conditions.

Definition. The equilibrium state $\mathbf{w} = 0$ is said to be Lyapunov unstable in the sense of the norm (3.18), if a number $\varepsilon > 0$ exists, that for any $\delta > 0$ a solution $\mathbf{w}(x, t)$ exists whose initial value obeys the inequality $|\mathbf{w}(x, 0)| < \delta$, while for a certain value $t_1 > 0$ the inequality $|\mathbf{w}(x, t_1)| \ge \varepsilon$ holds.

Such a definition is a special case of the definition of instability.[1,2] Instability of the equilibrium position of a one-dimensional non-linearly elastic body will be considered below first of all in accordance with this definition, and then also according to somewhat different definition, a special case of a more general Movchan definition.[3,4] We emphasize that when studying instability, it will always be assumed that for any initial state

$$\mathbf{w}(x, 0) = \begin{Vmatrix} u(x, 0) \\ \upsilon(x, 0) \end{Vmatrix}$$

satisfying the boundary conditions and the assumed conditions of smoothness, at least one classical solution of system (1.11) exists.

Theorem. *(Of instability by linear approximation) Let the linear operator A, specified by the first equality of* (1.10), *and setting the potential energy of the one-dimensional linearly elastic body with fixed ends, subject to a linear distributed external force, have at least one negative eigenvalue. Then the equilibrium position of the corresponding* *non-linearly elastic body, with a non-linear external force (the zero solution of the non-linear system of equations of motion (1.11)) is Lyapunov unstable in the sense of the norm (3.18).*

Note that the hypothesis of the theorem, which concerns the operator A, in terms of the potential energy for the linearized system (equal to $u\cdot Au/2$) implies the presence of negative values of this energy for some admissible displacement fields (vanishing at $x=0$; l).

Proof. We introduce the functional

$$\eta\{\mathbf{w}\} \equiv \mathbf{w}\bullet\mathbf{w} + \Psi\{u\} = u\cdot Bu + \upsilon\cdot\upsilon + \Psi\{u\} = 2e\{\mathbf{w}\} + u\cdot(B-A)u, \tag{4.1}$$

which will be referred to as the Lyapunov functional, although its properties are quite different from those of the conventional Lyapunov functional (function) used in theorems on instability.

It is almost obvious (and will be shown rigorously) that the term $\Psi\{u\}$ is as small as desired compared to $\mathbf{w}\cdot\mathbf{w}$ in a sufficiently small sphere $S(\bar{\delta})$, specified by the inequality

$$|\mathbf{w}| < \bar{\delta}. \tag{4.2}$$

Indeed, if $\mathbf{w}\neq 0$ while $u=0$, then $\Psi\{u\}=0$ and $\Psi\{u\}/\mathbf{w}\cdot\mathbf{w}=0$. If $u\neq 0$, then $u\cdot Bu>0$ and

$$\begin{aligned}\frac{|\Psi\{u\}|}{\mathbf{w}\bullet\mathbf{w}} &= \frac{|\Psi\{u\}|}{u\cdot Bu}\frac{u\cdot Bu}{\mathbf{w}\bullet\mathbf{w}} \le \frac{|\Psi\{u\}|}{u\cdot Bu} \le \frac{|\Psi\{u\}|}{\underline{C}\langle u_x^2\rangle} \le \\ &\le \frac{C_1}{(2+\sigma_1)\underline{C}}\frac{\langle |u_x|^{2+\sigma_1}\rangle}{\langle u_x^2\rangle} + \frac{C_2}{(2+\sigma_2)\underline{C}}\frac{\langle |u|^{2+\sigma_2}\rangle}{\langle u_x^2\rangle} \le \frac{C_1 l}{(2+\sigma_1)\underline{C}}\sup|u_x|^{\sigma_1} + \frac{C_2 l^{2+\sigma_2}}{(2+\sigma_2)\underline{C}}\sup|u_x|^{\sigma_2}.\end{aligned} \tag{4.3}$$

By virtue of the definition (3.18)

$$\sup|u_x| \le \frac{|\mathbf{w}|}{\sqrt{\underline{C}l}} < \frac{\bar{\delta}}{\sqrt{\underline{C}l}}, \quad \frac{|\Psi\{u\}|}{\mathbf{w}\bullet\mathbf{w}} \le C_3|\mathbf{w}|^{\sigma_1} + C_4|\mathbf{w}|^{\sigma_2}. \tag{4.4}$$

Since, by assumption, σ_1 and σ_2 are positive, then for an arbitrarily small $\xi>0$ it is possible to choose $\bar{\delta} > 0$ such that in the sphere $S(\bar{\delta})$ the following inequalities hold

$$-\xi < \frac{\Psi\{u\}}{\mathbf{w}\bullet\mathbf{w}} < \xi \Leftrightarrow 1-\xi < \frac{\eta\{\mathbf{w}\}}{\mathbf{w}\bullet\mathbf{w}} < 1+\xi \tag{4.5}$$

$$\Leftrightarrow 1-\xi < 1-\frac{\xi}{1+\xi} < \frac{\mathbf{w}\bullet\mathbf{w}}{\eta\{\mathbf{w}\}} < 1+\frac{\xi}{1-\xi}, \tag{4.6}$$

from which it follows, in particular, that at all points of the sphere $S(\bar{\delta})$, apart from the centre, the quantity $\eta\{\mathbf{w}\}$ is positive and its logarithm exists.

Having introduced beforehand the functionals

$$\zeta_\eta\{\mathbf{w}\} = \frac{\mathbf{w}\bullet\mathcal{A}\mathbf{w}}{\eta\{\mathbf{w}\}} = \frac{\mathbf{w}\bullet\mathcal{A}^s\mathbf{w}}{\eta\{\mathbf{w}\}}, \quad \zeta\{\mathbf{w}\} = \frac{\mathbf{w}\bullet\mathcal{A}^s\mathbf{w}}{\mathbf{w}\bullet\mathbf{w}} \tag{4.7}$$

we determine the derivative of the quantity $(\ln\eta)/2$ by virtue of the equations of motion

$$\frac{1}{2}(\ln\eta)^{\cdot} = \frac{u\cdot(B-A)\upsilon}{\eta\{\mathbf{w}\}} = \zeta_\eta\{\mathbf{w}\} = \zeta\{\mathbf{w}\}\frac{\mathbf{w}\bullet\mathbf{w}}{\eta\{\mathbf{w}\}}. \tag{4.8}$$

Here we have used the constancy of the total energy of a body and the equality

$$\mathbf{w}\cdot\mathcal{A}\mathbf{w} = u\cdot(B-A)\upsilon,$$

which follows from relations (3.1), (3.10) and the symmetry of the operators A and B.

According to equalities (3.17)

$$\mathcal{A}^s = \mathcal{A}^{s-} + \mathcal{A}^{s0} = \mathcal{A}^- + \mathcal{A}^{s0}, \tag{4.9}$$

i.e., in the subspace $\mathcal{W}^-$ the actions of the operators $\mathcal{A}^s$ and $\mathcal{A}$ are identical, in the subspace $\mathcal{W}^+$ the operator $\mathcal{A}^s$ is zero, while in the subspace $\mathcal{W}^0$ it acts in accordance with equality (3.15) as follows:

$$\mathcal{A}^s\begin{Vmatrix} u^0 \\ v^0 \end{Vmatrix} = \frac{1}{2}\begin{Vmatrix} v^0 \\ b_0u^0 \end{Vmatrix}.$$

We will find the values of the functional $\zeta_\eta\{\mathbf{w}\}$ on the vectors $\mathbf{w}_1^+$, lying in the sphere $S(\bar{\delta})$ and collinear with the vector $\boldsymbol{\gamma}_1^+$ (the latter is an eigenvector of $\mathcal{A}$ and $\mathcal{A}^s$ with eigenvalue $\alpha_1 = \sqrt{-a_1}$):

$$\zeta_\eta\{\mathbf{w}_1^+\} = \alpha_1\frac{\mathbf{w}_1^+\bullet\mathbf{w}_1^+}{\eta\{\mathbf{w}_1^+\}} > \alpha_1(1-\xi). \tag{4.10}$$

Thus, in a sufficiently small sphere $S(\bar{\delta})$ the values of $\zeta_\eta\{\mathbf{w}_1^+\}$ differ as little as desired from α_1 and have a lower bound of $\alpha_1(1-\xi)$. We select a positive number $\tilde{\alpha}$ satisfying the inequality

$$0 < \tilde{\alpha} < \alpha_1(1-\xi) \tag{4.11}$$

and call the set of non-zero vectors $\mathbf{w}$ a pseudocone $\Omega^p(\tilde{\alpha})$, for which

$$\tilde{\alpha} < \zeta_\eta\{\mathbf{w}\}. \tag{4.12}$$

From relations (4.10) and (4.11) it is clear that the intersection of the pseudocone $\Omega^p(\tilde{\alpha})$ and the sphere $S(\bar{\delta})$ is not empty. It is obvious that

$$\zeta_\eta\{\mathbf{w}\} = 0 < \tilde{\alpha}, \quad \mathbf{w}\in\mathcal{W}^+. \tag{4.13}$$

Due to the continuity of this functional, at the intersection of the pseudocone and the sphere, it takes all possible values of α obeying the inequalities $\tilde{\alpha} < \alpha \le \alpha_1(1-\xi)$; moreover, the boundary of the pseudocone $\partial\Omega^p(\tilde{\alpha})$ is the totality of $\mathbf{w} = 0$ and those non-zero vectors $\mathbf{w}$, for which

$$\zeta_\eta\{\mathbf{w}\} = \tilde{\alpha}. \tag{4.14}$$

The boundary of the pseudocone $\partial\Omega^p(\tilde{\alpha})$ partitions the sphere $S(\bar{\delta})$ into two sets:

$$\Omega^p(\tilde{\alpha})\cap S(\bar{\delta}) \text{ and } S(\bar{\delta})\backslash(\Omega^p(\tilde{\alpha})\cap S(\bar{\delta})).$$

If a continuous curve emanates from the first one and for some value of t, without leaving the sphere, enters the second one, then it inevitably crosses the boundary $\partial\Omega^p(\tilde{\alpha})$ (passing through the vertex $\mathbf{w} = 0$ will be excluded in what follows).

We will first show that if the solution $\mathbf{w}(t)$ takes a non-zero initial value $\mathbf{w}(0)\in\Omega^p(\tilde{\alpha})\cap S(\bar{\delta})$ and does not leave the pseudocone, then for some $t = t_1$ it reaches the boundary of the sphere $|\mathbf{w}| = \bar{\delta}/2$. Suppose the contrary, namely that $\mathbf{w}(t)\in S(\bar{\delta}/2)$, $\forall t$. Hence,

$$\frac{1}{2}(\ln\eta)^{\cdot} > \tilde{\alpha} \Rightarrow \eta(t) > \eta(0)e^{2\tilde{\alpha}t}$$

$$(1-\xi)\eta\{\mathbf{w}\} < \mathbf{w}\bullet\mathbf{w} \le |\mathbf{w}|^2 \Rightarrow (1-\xi)\eta(0)e^{2\tilde{\alpha}t} < |\mathbf{w}(t)|^2.$$

Note that the above-mentioned impossibility for the solution considered to pass through the vertex of pseudocone therefore follows.

If the solution for all $t > 0$ remained within the sphere $S(\bar{\delta}/2)$, then the inequality

$$(1-\xi)\eta(0)e^{2\tilde{\alpha}t} < \bar{\delta}^2/4,$$

would always be valid, which is impossible. Then, for a certain value of $t = t_1$

$$|\mathbf{w}(t_1)| = \bar{\delta}/2. \tag{4.15}$$

However, this just means instability in Lyapunov's sense, since it is possible to take a $\mathbf{w}(0) \neq 0$, for which

$$0 < |\mathbf{w}(0)| < \delta, \tag{4.16}$$

where δ is as small a number as desired. Moreover,

$$\eta(0) > (1-\xi)\mathbf{w}(0) \bullet \mathbf{w}(0) > 0. \tag{4.17}$$

Taking $\varepsilon = \bar{\delta}/2$, we obtain complete agreement with the definition of instability.

Thus, it remains to prove (and this is the main thing), that the solution which remains within a sufficiently small sphere, will in fact not leave at least one suitable pseudocone, which will imply that the set $\Omega^p(\tilde{\alpha}) \cap S(\bar{\delta})$ is a sector.[11]

First we will show that the part of boundary of the pseudocone $\Omega^p(\tilde{\alpha})$, lying within a sufficiently small sphere, is located between the boundaries of two cones $\Omega(\tilde{\alpha}(1-\xi))$ and $\Omega(\tilde{\alpha}(1+\xi))$ which differ by as little as desired; by the cone $\Omega(\beta)$ we mean here a set of such non-zero vectors $\mathbf{w}$, for which

$$\beta < \zeta\{\mathbf{w}\}. \tag{4.18}$$

The boundary $\partial\Omega(\beta)$ for $\mathbf{w} \neq 0$ is specified by the equality

$$\zeta\{\mathbf{w}\} = \beta. \tag{4.19}$$

In what follows it will always be assumed, that $\mathbf{w} \neq 0$. Let $\mathbf{w} \in \partial\Omega^p(\tilde{\alpha})$.

Then

$$\zeta\{\mathbf{w}\} = \zeta_\eta\{\mathbf{w}\}\frac{\eta\{\mathbf{w}\}}{\mathbf{w} \bullet \mathbf{w}} = \tilde{\alpha}\frac{\eta\{\mathbf{w}\}}{\mathbf{w} \bullet \mathbf{w}}$$

and in the sphere $S(\bar{\delta})$, by virtue of the second inequality (4.5), we have

$$\tilde{\alpha}(1-\xi) \le \zeta\{\mathbf{w}\} \le \tilde{\alpha}(1+\xi) \Leftrightarrow \mathbf{w} \in \bar{\Omega}(\tilde{\alpha}(1-\xi)) \backslash \Omega(\tilde{\alpha}(1+\xi)). \tag{4.20}$$

If the trajectory $\mathbf{w}(t)$ emanating from the interior of the pseudocone $\Omega^p(\tilde{\alpha})$, were to arrive at a time t_2 at its boundary, then the quantity $\zeta_\eta\{\mathbf{w}\}$ at this time would not increase and, hence, its time derivative, by virtue of the system (1.11), could only be non-positive (this follows from Taylor's formula for $\zeta_\eta\{\mathbf{w}\}$ in the vicinity of $t = t_2$). If the analysis reveals that at any point $\mathbf{w} \in \partial\Omega^p(\tilde{\alpha}) \cap S(\bar{\delta})$ the derivative by virtue of the system is positive, then this means that a trajectory emanating from the interior of the pseudocone cannot reach its boundary and remains inside it.

We will obtain the time derivative of the quantity $\zeta_\eta\{\mathbf{w}\}$ by virtue of system (1.11) and we will show that positive values of $\tilde{\alpha}$ exist, for which this derivative is positive on the part of the pseudocone boundary $\partial\Omega^p(\tilde{\alpha}) \cap S(\bar{\delta})$:

$$\begin{aligned}(\zeta_\eta\{\mathbf{w}\})^{\bullet} &= \frac{2\dot{\mathbf{w}} \bullet \mathcal{A}^s\mathbf{w}}{\eta\{\mathbf{w}\}} - \frac{(\mathbf{w} \bullet \mathcal{A}^s\mathbf{w})\dot{\eta}}{\eta^2} = \\ &= 2\left(\frac{(\mathcal{A}\mathbf{w} + \mathbf{f}\{\mathbf{w}\}) \bullet \mathcal{A}^s\mathbf{w}}{\eta\{\mathbf{w}\}} - (\zeta_\eta\{\mathbf{w}\})^2\right) = 2\frac{\mathbf{w} \bullet \mathbf{w}}{\eta\{\mathbf{w}\}}\left(\frac{(\mathcal{A}\mathbf{w} + \mathbf{f}\{\mathbf{w}\}) \bullet \mathcal{A}^s\mathbf{w}}{\mathbf{w} \bullet \mathbf{w}} - \frac{\eta\{\mathbf{w}\}}{\mathbf{w} \bullet \mathbf{w}}\tilde{\alpha}^2\right).\end{aligned} \tag{4.21}$$

By virtue of inequalities (4.6), $\mathbf{w}\cdot\mathbf{w}/\eta$ is close to unity in the sphere $S(\bar{\delta})$, and so the sign of the derivative is determined by the sign of the expression in brackets in the last of equalities of (4.21). This expression (using equality (4.19)) can be represented as

$$Y\{\mathbf{w}\} \equiv \frac{(\mathcal{A}\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w}}{\mathbf{w} \bullet \mathbf{w}} - \beta^2 + \frac{\mathbf{f}\{\mathbf{w}\} \bullet \mathcal{A}^s\mathbf{w}}{\mathbf{w} \bullet \mathbf{w}} - \tilde{\alpha}^2\left(1 - \frac{\eta\{\mathbf{w}\}}{\mathbf{w} \bullet \mathbf{w}}\right)\frac{\eta\{\mathbf{w}\}}{\mathbf{w} \bullet \mathbf{w}}. \tag{4.22}$$

The modulus of the last term is bounded by the quantity $\tilde{\alpha}^2\xi(1+\xi)$ and so is as small as desired in a sufficiently small sphere $S(\bar{\delta})$. We will show that this also holds regarding the penultimate term, from which it will follow that the sign of $Y\{\mathbf{w}\}$ is determined by the sign of the sum of first two terms.

Note, that if $\mathbf{w} = \begin{Vmatrix} 0 \\ v \end{Vmatrix}$, then $\mathbf{f}\{\mathbf{w}\} = 0$ and $\mathbf{f}\{\mathbf{w}\} \cdot \mathscr{A}^S\mathbf{w} = 0$, while $\mathbf{w}\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{v} > 0$; hence, the penultimate term vanishes. If $\mathbf{w} = \begin{Vmatrix} u \\ v \end{Vmatrix}$, then

$$\mathbf{f}\{\mathbf{w}\} \bullet \mathscr{A}^s\mathbf{w} = f\{u\} \cdot \left(-A^- + \frac{1}{2}b_0 I^0\right)u =$$

$$= \frac{1}{2}\left\langle\left(\left(\frac{\partial\psi}{\partial u_x}\right)_x + \frac{\partial\varphi}{\partial u}\right)\left(\sum_{i=1}^{m}(-a_i)g_i\langle\mu g_i u\rangle + \sum_{k=m+1}^{m+m'}\frac{b_0}{2}g_k\langle\mu g_k u\rangle\right)\right\rangle. \tag{4.23}$$

We will take for b_0 the preliminary inequality

$$0 < b_0 \le |a_1|. \tag{4.24}$$

Taking into account the fact that all of $g_i(x)$ vanish at the ends of the interval $[0, l]$, we have

$$\left\langle\left(\frac{\partial\psi}{\partial u_x}\right)_x g_i\right\rangle = -\left\langle\frac{\partial\psi}{\partial u_x}g_{ix}\right\rangle$$

$$|\mathbf{f} \bullet \mathscr{A}^s\mathbf{w}| \le \frac{|a_1|}{2}\sum_{i=1}^{m+m'}\left|\left\langle\frac{\partial\psi}{\partial u_x}g_{ix}\right\rangle\langle\mu g_i u\rangle\right| + \frac{|a_1|}{2}\sum_{i=1}^{m+m'}\left|\left\langle\frac{\partial\varphi}{\partial u}g_i\right\rangle\langle\mu g_i u\rangle\right|$$

$$|\langle\mu g_i u\rangle| \le \sqrt{\bar{\mu}}\, l\sqrt{\langle u_x^2\rangle} \tag{4.25}$$

$$\left|\left\langle\frac{\partial\varphi}{\partial u}g_i\right\rangle\right| \le C_2\left|\langle|u|^{1+\sigma_2}g_i\rangle\right| \le C_2\sup|u|^{1+\sigma_2}|\langle g_i\rangle| \le C_2 l^{(1+\sigma_2)/2}\langle u_x^2\rangle^{(1+\sigma_2)/2}\sqrt{l\langle g_i^2\rangle} \le$$

$$\le C_2 l^{(1+\sigma_2)/2}(l^{1/2}/\underline{\mu}^{1/2})\langle u_x^2\rangle^{(1+\sigma_2)/2} = \frac{C_2 l^{1+\sigma_2/2}}{\underline{\mu}^{1/2}}\langle u_x^2\rangle^{(1+\sigma_2)/2}$$

$$\left|\left\langle\frac{\partial\varphi}{\partial u}g_i\right\rangle\langle\mu g_i u\rangle\right| \le C_2 l^{2+\sigma_2/2}\left(\frac{\bar{\mu}}{\underline{\mu}}\right)^{1/2}\langle u_x^2\rangle\sup|u_x|^{\sigma_2}$$

$$\left|\left\langle\frac{\partial\psi}{\partial u_x}g_{ix}\right\rangle\right| \le C_1\left|\langle|u_x|^{1+\sigma_1}g_{ix}\rangle\right| \le C_1\sup|u_x|^{\sigma_1}\langle u_x^2\rangle^{1/2}\langle g_{ix}^2\rangle^{1/2}$$

$$0 \ge a_i = g_i \cdot Ag_i = \langle pg_{ix}^2\rangle - \langle qg_i^2\rangle \ge \underline{p}\langle g_{ix}^2\rangle - \frac{\bar{q}}{\underline{\mu}}\langle\mu g_i^2\rangle = \underline{p}\langle g_{ix}^2\rangle - \frac{\bar{q}}{\underline{\mu}} \Rightarrow \langle g_{ix}^2\rangle \le \frac{\bar{q}}{\underline{\mu}\underline{p}}$$

$$\left|\left\langle\frac{\partial\psi}{\partial u_x}g_{ix}\right\rangle\langle\mu g_i u\rangle\right| \le C_1 l\bar{\mu}^{1/2}\langle g_{ix}^2\rangle^{1/2}\langle u_x^2\rangle\sup|u_x|^{\sigma_1} \le C_1 l\left(\frac{\bar{\mu}\bar{q}}{\underline{\mu}\underline{p}}\right)^{1/2}\langle u_x^2\rangle\sup|u_x|^{\sigma_1}.$$

Carrying out the summation in (4.25) and taking into account the first inequality of (4.4), we finally obtain

$$|\mathbf{f} \bullet \mathscr{A}^s\mathbf{w}| \le \langle u_s^2\rangle(C_5|\mathbf{w}|^{\sigma_1} + C_6|\mathbf{w}|^{\sigma_2}). \tag{4.26}$$

Thus,

$$\frac{|\mathbf{f} \bullet \mathscr{A}^s\mathbf{w}|}{\mathbf{w} \bullet \mathbf{w}} \le \frac{|\mathbf{f} \bullet \mathscr{A}^s\mathbf{w}|}{u \cdot Bu} \le \frac{|\mathbf{f} \bullet \mathscr{A}^s\mathbf{w}|}{\underline{C}\langle u_x^2\rangle} \le \frac{C_5}{\underline{C}}|\mathbf{w}|^{\sigma_1} + \frac{C_6}{\underline{C}}|\mathbf{w}|^{\sigma_2}. \tag{4.27}$$

The right-hand side of this inequality is as small as desired in the sphere $S(\bar{\delta})$ for sufficiently small $\bar{\delta}$.

It remains to prove that positive values of $\tilde{\alpha}$ exist, such that for $\mathbf{w} \neq 0$, $\mathbf{w} \in \partial\Omega^p(\tilde{\alpha}) \cap S(\bar{\delta})$ the following inequality holds

$$\frac{(\mathcal{A}\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w}}{\mathbf{w} \bullet \mathbf{w}} > \beta^2. \tag{4.28}$$

We will first prove this for the simpler non-degenerate case when the operator A has no zero eigenvalues, and the inequality will then be proved for the degenerate case, too. If the operator A is non-degenerate, then the subspaces U° and $\mathcal{W}^\circ$ are empty, $B = -A^- + A^+$, and the operators $\mathcal{A}^a$ and $\mathcal{A}^s$ are non-zero in mutually orthogonal subspaces, which are invariant; therefore

$$(\mathcal{A}^a\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w} = 0 \Rightarrow (\mathcal{A}\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w} = (\mathcal{A}^s\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w}. \tag{4.29}$$

The operator $\mathcal{A}^s$ has $2m$ non-zero eigenvalues α_1, $(-\alpha_1)$, ..., α_m, $(-\alpha_m)$, where $\alpha_i = \sqrt{-a_i}$, with eigenvectors $\boldsymbol{\gamma}_i^+$, $\boldsymbol{\gamma}_i^-$, and is identically equal to zero in the infinite-dimensional subspace $\mathcal{W}^+$. We select $\tilde{\alpha}$ and $\bar{\delta}$ so, that the interval $[\tilde{\alpha}(1-\xi), \tilde{\alpha}(1+\xi)]$ lies inside the interval whose ends are two neighboring eigenvalues differing from one another; for example,

$$\alpha_2 < \tilde{\alpha}(1-\xi) < \tilde{\alpha}(1+\xi) < \alpha_1. \tag{4.30}$$

If there is only one negative eigenvalue $a_1 < 0$ (and correspondingly, one $\alpha_1 = \sqrt{-a_1} > 0$), then suppose for example, that

$$0 < \alpha_1/2 < \tilde{\alpha}(1-\xi) < \tilde{\alpha}(1+\xi) < \alpha_1.$$

Here, by virtue of inequalities (4.20) we have

$$\zeta\{\mathbf{w}\} = \beta \Rightarrow \tilde{\alpha}(1-\xi) \le \beta \le \tilde{\alpha}(1+\xi) \Rightarrow \alpha_2 < \beta < \alpha_1, \tag{4.31}$$

i.e., the quantity β differs from any of the eigenvalues of the operator $\mathcal{A}^s$. Suppose

$$\mathbf{h} = \mathcal{A}^s\mathbf{w} - \beta\mathbf{w} \neq 0 \Rightarrow \mathbf{h} \bullet \mathbf{w} = \mathbf{w} \bullet \mathcal{A}^s\mathbf{w} - \beta\mathbf{w} \bullet \mathbf{w} = 0 \tag{4.32}$$

$$\begin{gathered}(\mathcal{A}^s\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w} = (\beta\mathbf{w} + \mathbf{h}) \bullet (\beta\mathbf{w} + \mathbf{h}) = \beta^2\mathbf{w} \bullet \mathbf{w} + \mathbf{h} \bullet \mathbf{h} \Rightarrow \\ \Rightarrow \frac{(\mathcal{A}^s\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w}}{\mathbf{w} \bullet \mathbf{w}} = \beta^2 + \frac{\mathbf{h} \bullet \mathbf{h}}{\mathbf{w} \bullet \mathbf{w}}.\end{gathered} \tag{4.33}$$

We will represent $\mathbf{w}$ as the sum of eigenvectors (which are mutually orthogonal)

$$\mathbf{w} = \mathbf{w}_1^+ + \mathbf{w}_1^- + \ldots + \mathbf{w}_m^+ + \mathbf{w}_m^- + \mathbf{w}^\perp, \tag{4.34}$$

We have

$$\mathbf{h} = (\alpha_1 - \beta)\mathbf{w}_1^+ - (\alpha_1 + \beta)\mathbf{w}_1^- + \ldots + (\alpha_m - \beta)\mathbf{w}_m^+ - (\alpha_m + \beta)\mathbf{w}_m^- - \beta\mathbf{w}^\perp$$

$$\begin{gathered}\mathbf{h} \bullet \mathbf{h} = (\alpha_1 - \beta)^2\mathbf{w}_1^+ \bullet \mathbf{w}_1^+ + (\alpha_1 + \beta)^2\mathbf{w}_1^- \bullet \mathbf{w}_1^- + \ldots + (\alpha_m - \beta)^2\mathbf{w}_m^+ \bullet \mathbf{w}_m^+ + \\ + (\alpha_m + \beta)^2\mathbf{w}_m^- \bullet \mathbf{w}_m^- + \beta^2\mathbf{w}^\perp \bullet \mathbf{w}^\perp \ge \min((\alpha_i - \beta)^2, \beta^2)\mathbf{w} \bullet \mathbf{w}.\end{gathered}$$

Thus,

$$\frac{(\mathcal{A}^s\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w}}{\mathbf{w} \bullet \mathbf{w}} - \beta^2 \ge \min((\alpha_i - \beta)^2, \beta^2). \tag{4.35}$$

The right-hand side of inequality (4.35) is positive and finite. Hence, for sufficiently small $\bar{\delta}$ and, so, sufficiently small ξ, jointly with it, the derivative (4.21) is strictly positive, too. Hence, the trajectories emanating from non-zero points of intersection of regions $\Omega^p(\tilde{\alpha})$ and $S(\bar{\delta})$, while remaining in the region $S(\bar{\delta})$, do not leave the region $\Omega^p(\tilde{\alpha})$, but they then reach the boundary $\partial S(\bar{\delta}/2)$, which just implies the instability with respect to the norm $|\mathbf{w}|$.

We now turn to the case of a degenerate operator A. In this case, $B = -A^- + b_0 I^0 + A^+$ (3.11), where the positive number b_0 is now restricted only by the inequality $b_0 \le |a_1|$ (4.24), but in principle it can be taken to be fairly small (although finite), which will be done in what follows. Note that the lower bound for the number b_0 is stipulated by the fact that it occurs in the lower estimate for the form $u \cdot Bu$ (2.10) via the equality (2.7).

In the case when the operators $\mathcal{A}^s$ and $\mathcal{A}^a$ are degenerate they are geared in the subspace $\mathcal{W}^0$:

$$(\mathcal{A}\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w} = (\mathcal{A}^s\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w} + (\mathcal{A}^a\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w} = (\mathcal{A}^s\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w} + v^0 \cdot Bv^0/4 - b_0^2u^0 \cdot u^0/4 =$$
$$= (\mathcal{A}^s\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w} - b_0(u^0 \cdot Bu^0 + v^0 \cdot v^0)/4 + b_0 v^0 \cdot v^0/2 \ge (\mathcal{A}^s\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w} - b_0\mathbf{w} \bullet \mathbf{w}/4$$

$$\beta_0 \equiv \sqrt{b_0}/2 \Rightarrow (\mathcal{A}\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w} \ge (\mathcal{A}^s\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w} - \beta_0^2\mathbf{w} \bullet \mathbf{w}.$$

The right-hand side of the last inequality is investigated in the same way in the non-degenerate case, but the eigenvalues of the operator $\mathcal{A}^s$ in the subspace $\mathcal{W}^0$ must be obtained. It can be verified, that each of the vectors g_i, $i = m+1, \ldots, m+m'$ (that form an orthonormal basis in U^0) generates a pair of eigenvectors of the operator $\mathcal{A}^S$ with eigenvalues β_0 and $-\beta_0$:

$$\mathcal{A}^s(\mathbf{g}_i^u \pm 2\beta_0\mathbf{g}_i^v) = \pm\beta_0(\mathbf{g}_i^u \pm 2\beta_0\mathbf{g}_i^v), \quad i = m+1, \ldots, m+m'.$$

Denoting the linear combination of eigenvectors with the eigenvalues β_0 in the expansion of $\mathbf{w}$ by $\mathbf{w}_0^+$, and the analogous combination for the eigenvalue $-\beta_0$ by $\mathbf{w}_0^-$, we obtain

$$\mathbf{h} = \mathcal{A}^s\mathbf{w} - \beta\mathbf{w} = (\alpha_1 - \beta)\mathbf{w}_1^+ - (\alpha_1 + \beta)\mathbf{w}_1^- + \ldots + (\alpha_m - \beta)\mathbf{w}_m^+ -$$
$$- (\alpha_m + \beta)\mathbf{w}_m^- + (\beta_0 - \beta)\mathbf{w}_0^+ - (\beta_0 + \beta)\mathbf{w}_0^- - \beta\mathbf{w}^\perp$$

$$\mathbf{h} \bullet \mathbf{h} \ge \min((\alpha_i - \beta)^2, (\beta_0 - \beta)^2, \beta^2)\mathbf{w} \bullet \mathbf{w}$$

Hence

$$\begin{aligned}(\mathcal{A}\mathbf{w}) \bullet \mathcal{A}^s\mathbf{w} &\ge \beta^2\mathbf{w} \bullet \mathbf{w} + \mathbf{h} \bullet \mathbf{h} - \beta_0^2\mathbf{w} \bullet \mathbf{w} \ge \\ &\ge (\beta^2 + \min((\alpha_i - \beta)^2, (\beta_0 - \beta)^2, \beta^2) - \beta_0^2)\mathbf{w} \bullet \mathbf{w}.\end{aligned} \tag{4.36}$$

We recall that β differs by as little as desired from $\tilde{\alpha}$ (4.31), and it needs to be shown that $\tilde{\alpha}$ and β_0 can be chosen so that they are positive and finite, whereas the difference $\min\left((\alpha_i - \beta)^2, \left(\beta_0 - \beta\right)^2, \beta^2\right) - \beta_0^2$ is also positive and finite. Consider two cases: (1) $\tilde{m} = 1$ and (2) $\tilde{m} > 1$, where by $\tilde{m}$ we mean the number of distinct negative eigenvalues of the operator A (and an equal number of distinct values of α_i).

Case 1. There is only $\alpha_1 > 0$. Suppose

$$\beta_0 = \frac{\alpha_1}{7}, \quad \tilde{\alpha} = \frac{4}{7}\alpha_1. \tag{4.37}$$

Then

$$\begin{aligned}&\frac{3}{7}\alpha_1 < \beta < \frac{5}{7}\alpha_1 \Rightarrow \alpha_1 - \beta > \frac{2}{7}\alpha_1, \quad \beta - \beta_0 > \frac{2}{7}\alpha_1 \Rightarrow \\ &\Rightarrow \min((\alpha_1 - \beta)^2, (\beta_0 - \beta)^2, \beta^2) - \beta_0^2 > 3\left(\frac{\alpha_1}{7}\right)^2.\end{aligned} \tag{4.38}$$

Case 2. There are at least two distinct neighboring eigenvalues α_1 and α_2, furthermore

$$0 < \alpha_2 < \alpha_1. \tag{4.39}$$

Suppose

$$\beta_0 = \min\left(\alpha_2, \frac{\alpha_1 - \alpha_2}{6}\right), \quad \tilde{\alpha} = \frac{\alpha_1 + \alpha_2}{2}. \tag{4.40}$$

Then

$$|\alpha_1-\beta|>\frac{\alpha_1-\alpha_2}{3},\quad |\alpha_2-\beta|>\frac{\alpha_1-\alpha_2}{3},\quad \beta>\frac{\alpha_1-\alpha_2}{3},\quad |\beta_0-\beta|>\frac{\alpha_1-\alpha_2}{3}$$
$$\beta_0\le\frac{\alpha_1-\alpha_2}{6}\Rightarrow \min((\alpha_1-\beta)^2,(\alpha_2-\beta)^2,(\beta_0-\beta)^2,\beta^2)-\beta_0^2>(\alpha_1-\alpha_2)^2/12. \tag{4.41}$$

Thereby the possibility of choosing appropriate values of the numbers $\tilde{\alpha}$ and β_0 in both cases is shown.

This completes the proof of the instability of the system considered with respect to the norm $|\mathbf{w}|$ (3.18). However, the norm (3.18) contains both the "strain" term (the first term) and the velocity term, namely, the kinetic energy. Nevertheless it is traditional in the theory of elasticity and in the mechanics of solids in general, that the displacements and strains are of the main interest; measures of the deviation of a system from the equilibrium position, investigated for stablility or instability are usually related to them. Therefore it would be desirable to prove that the solution $\mathbf{w}(t)$ increases not only in the norm (3.18), but also in some strain norm, not dependent on the velocities. In Movchan's terminology this corresponds to instability with respect to two metrics, and this kind of instability will be derived in the next section from the already proven instability with respect to the norm $|\mathbf{w}|$.

5. Definition of instability with respect to two norms and the proof of its presence

Specifying for the problem in hand the definition of instability with respect to two metrics,[4] we state the following definition of instability with respect to two norms.

Definition. The zero solution of system (1.11) is said to be unstable with respect to two norms, if $\varepsilon>0$ can be found, such that for any $\delta>0$ a solution $\mathbf{w}(t)$ exists, obeying at the initial instant of time the inequality

$$|\mathbf{w}(0)|<\delta, \tag{5.1}$$

and at some instant t_1 the inequality

$$\sup|u_x(x,t_1)|\ge\varepsilon. \tag{5.2}$$

Note that it was proved above that a solution $\mathbf{w}(t)$ exists, which reaches the boundary of a sphere $S(\bar{\delta}/2)$ and at the same time does not leave the pseudocone $\Omega^p(\tilde{\alpha})$, $0<\tilde{\alpha}<\alpha_1$. By virtue of the inequalities (4.20) in combination with the smallness of the sphere $S(\bar{\delta}/2)$ this means that $\mathbf{w}(t)$ does not leave the cone $\Omega(\tilde{\alpha}(1-\xi))$, where ξ is a number as small as desired:

$$\mathbf{w}\bullet\mathcal{A}^s\mathbf{w}\ge\tilde{\alpha}(1-\xi)\mathbf{w}\bullet\mathbf{w}\equiv\tilde{\beta}\mathbf{w}\bullet\mathbf{w}=\tilde{\beta}(u\cdot Bu+\upsilon\cdot\upsilon)\ge\tilde{\beta}\upsilon\cdot\upsilon.$$

Hence it follows, that

$$\upsilon\cdot\upsilon\le\frac{1}{\tilde{\beta}}\mathbf{w}\bullet\mathcal{A}^s\mathbf{w}=\frac{1}{\tilde{\beta}}\upsilon\cdot(B-A)u$$

$$(\upsilon\cdot\upsilon)^2\le\frac{1}{\tilde{\beta}^2}(\upsilon\cdot(B-A)u)^2\le\frac{1}{\tilde{\beta}^2}(\upsilon\cdot\upsilon)((B-A)u)\cdot(B-A)u\le$$
$$\le\frac{4\alpha_1^2}{\tilde{\beta}^2}(\upsilon\cdot\upsilon)u\cdot Bu\Rightarrow\upsilon\cdot\upsilon\le\frac{4\alpha_1^2}{\tilde{\beta}^2}u\cdot Bu.$$

By virtue of the estimate (2.10), in the cone $\Omega(\tilde{\beta})$ we have

$$\upsilon\cdot\upsilon\le\frac{4\alpha_1^2}{\tilde{\beta}^2}\bar{C}\langle u_x^2\rangle\le\frac{4\alpha_1^2}{\tilde{\beta}^2}\bar{C}l\sup(u_x^2),\quad |\mathbf{w}|^2\le\left(1+\frac{4\alpha_1^2}{\tilde{\beta}^2}\right)\bar{C}l\sup(u_x^2).$$

It is obvious, that, if we take

$$\varepsilon = \left[\left(1+\frac{4\alpha_1^2}{\tilde{\beta}^2}\right)\bar{C}l\right]^{-1/2}\frac{\bar{\delta}}{2},$$

then at the time when the solution reaches the sphere boundary $\partial S(\bar{\delta}/2)$, i.e. when the equality $|\mathbf{w}(t_1)| = \bar{\delta}/2$ is satisfied, inequality (5.2) will hold, which also denotes the instability with respect to two norms defined above.

6. Concluding remarks

The version of Persidskii's sector method used here differs from its published versions: as a sector in the phase space of a system we take the cone whose surface is specified by a fixed positive value of the characteristic exponent of growth for a certain norm of solutions. It is essential that this value be chosen so that it is different from any of the positive eigenvalues of the operator of linear approximation.

The instability with respect to two norms proved in the final analysis has the following clear mechanical interpretation: however small the initial disturbances in velocities, displacements or strains, among them there are those that, in the course of subsequent motion of the body, the strain attains a specified finite value, at least at some point of the body.

We would expect the method of proof suggested here to be quite suitable in principle for more general conservative continuous systems also (for example, for three-dimensional elastic bodies).

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